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Linear–state control problems and differential games: Deterministic and stochastic systems

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Abstract This paper concerns a class of *linear-state* optimal control problems and noncooperative differential games. Deterministic and stochastic systems are considered, as well as finite- and infinite-horizon problems. We give conditions under which these systems have *degenerate* feedback optimal controls so that the optimal control actions $a(t, x) \equiv a(t)$ are independent of the state variable x. As a consequence, open-loop and feedback (or Markov) optimal controls coincide, the value (or optimal objective) function is linear in the state x, and the *certainty equivalence principle* is satisfied.

Keywords Deterministic and stochastic optimal control · Deterministic and stochastic differential games · Maximum principle · Linear-state systems

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1 Introduction

Identifying optimal control problems and dynamic games with particular classes of optimal strategies has been an important research topic for many years. Midler [22] was, to the best of our knowledge, the first to note that a certain class of stochastic discrete-time *linear–state* control problems has "degenerate" feedback optimal controls, that is, optimal controls $a(t, x) \equiv a(t)$ independent of the state variable x, and the value (or optimal reward) function is linear in the state x. Moreover, the control problem satisfies the *certainty equivalence principle* [14,28] so that the optimal control for the associated deterministic problem (with $\xi_t \equiv 0$ in (2) below) is also optimal for the original stochastic problem.

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More explicitly, Midler [22] considered the following optimal control problem (OCP): maximize the expected performance index

$$J(\pi, x) := E\left[\sum_{t=0}^{T} r(t, x_t, a_t)\right] \tag{1}$$

over all controls $\pi = \{a_t\}$ in the control space $A \subset \mathbb{R}^m$ given that the state process $x_t \in \mathbb{R}^n$ evolves according to the stochastic difference equation

$$x_{t+1} = F(t, x_t, a_t) + \xi_t, \ \ t = 0, 1, ..., T-1, \tag{2}$$

with a given, possibly random, initial condition x_0, where the ξ_t are independent random vectors. The key feature in Midler's OCP is that the stage reward r and the system function F are linear in the state x, i.e.,

$$r(t, x, a) := \langle r_1(t), x\rangle + r_2(t, a), \quad F(t, x, a) := F_1(t)x + F_2(t, a), \tag{3}$$

where $\langle a, b\rangle$ denotes the inner product of two n–vectors a, b. Then, under mild assumptions (introduced in Section 2, below), Midler uses the inductive dynamic programming (or *value iteration*) algorithm to show that the optimal control law is independent of the state. (This result is also valid for infinite–horizon problems; see [13], exercise 3.13.)

Similar results were obtained in the 1970s and 1980s for several classes of deterministic control problems and differential games; see [6,9,17,21,24], among others [15]. Many of these results were collected by Dockner et al. in [7] (see also Chapter 7 in [8] or section 7.6 in [12]) and classified in several classes, such as, "state redundant", "state separable", and so on. The proof of these properties, however, were mostly indirect, that is, invoking results from other sources, such as the sufficient conditions in Stalford and Leitmann (1973). This brings us to our contributions in this paper.

We consider finite- and infinite-horizon, deterministic and stochastic continuous -time optimal control problems and differential games which are linear-state in the sense of (3) and give *direct proofs* that the corresponding optimal controls or Nash equilibria do not depend on the state variable. As a consequence, the associated value functions are linear in the state. As far as we can tell, the results for the stochastic differential case are new. Our proofs are mainly based on standard facts on linear systems. Midler's [22] inductive proof for discrete-time systems is not applicable in our context.

The remainder of the paper is organized as follows. Section 2 concerns finite- and infinite-horizon linear-state deterministic optimal control problems (OCPs). It also includes a summary of results on linear ordinary differential equations, which are used in several of the proofs. Section 3 is about linear-state (deterministic) differential games. Sections 4 and 5 extend to *stochastic* control problems and differential games the results in Sections 2 and 3, respectively. Section 6 presents several examples to illustrate our results and, finally, we conclude in Section 7 with some general observations on our results.

2 Deterministic optimal control problems: The finite-horizon case

In this section we first consider a finite–horizon optimal control problem (OCP) with dynamics and running reward function as in (3), i.e.,

$$\dot{x}(t) = F_1(t)x(t) + F_2(t, a(t)), \ \ x(0) = x_0, \tag{4}$$

$$r(t,x,a) = \langle r_1(t), x\rangle + r_2(t,a), \tag{5}$$

with $t \in [0,T]$, $r_1 \not\equiv 0$ and state and action spaces $X \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$, respectively. At the outset, we consider control functions $a(\cdot)$ in the set $\mathcal{A}[0,T]$ of piecewise continuous functions $a : [0,T] \to A$.

The OCP is to maximize the objective function (or performance index)

$$J(x_0, T, a(\cdot)) := \int_0^T r(t, x(t), a(t))\mathrm{d}t \tag{6}$$

over $\mathcal{A}[0,T]$, with r as in (5), subject to (4). In (6), $x(t) \equiv x(t; x_0, a(\cdot))$, $t \in [0,T]$, denotes the solution of (4). We will use the simpler notation $x(\cdot)$ instead of $x(\cdot; x_0, a(\cdot))$ whenever there is no risk of confusion.

To ensure that this OCP is well defined we will suppose the following.

Assumption 2.1 *Let $\varphi(t,x,a)$ be any of the functions r and F in* (4)-(5).

(a) *The function φ is uniformly continuous and there is a constant L such that $|\varphi(t,0,a)| \le L$ for all (t,a) in $[0,T] \times A$.*
(b) *For each $t \in [0,T]$, there are maximizers of the mappings $a \mapsto r_2(t,a)$ and $a \mapsto F_2(t,a)$.*

Under the Assumption 2.1(a), φ satisfies a Lipschitz condition in $x \in X$, i.e.,

$$|\varphi(t,x,a) - \varphi(t,x',a)| \le L|x - x'|$$

Therefore, the OCP (4)–(6) is well defined in the sense that (4) has a unique solution $x(\cdot; s, y, a(\cdot))$ for every initial condition (s,y) in $[0,T] \times X$ and $a(\cdot) \in \mathcal{A}[0,T]$, and also (6) is well defined. (See, for instance, Dockner et al. [8], Haurie et al. [12], or Yong and Zhou [29].) On the other hand, by the continuity in Assumption 2.1(a), sufficient conditions for Assumption 2.1(b) are well known. For instance, it suffices that A is compact, or that A is arbitrary but $a \mapsto r_2(t,a)$ and $a \mapsto F_2(t,a)$ are sup–compact for each $t \in [0,T]$, that is, for every $\alpha \in \mathbb{R}$, the set $\{a \in A : r_2(t,a) \ge \alpha\}$ is compact for each $t \in [0,T]$, and similarly for F_2.

2.1 **Summary of linear ODEs**

To state our results we need some facts on linear ODEs.

Let $\mathbb{R}^{n\times n}$ be the real linear space of $n \times n$ real matrices. Let $M : [c,d] \to \mathbb{R}^{n\times n}$ be a piecewise continuous function, and consider the following linear ODE:

$$\dot{x}(t) = M(t)x(t) \quad \forall t \in [c,d], \quad x(c) = \beta. \tag{7}$$

It is well known [18] that equation (7) has a unique solution given by

$$x(t) = \Phi(t,c)\beta \quad \forall t \in [c,d],$$

where $\Phi : [c,d] \times [c,d] \to \mathbb{R}^{n\times n}$ is the transition matrix generated by M, which is given by the *Peano-Baker series*

$$\Phi(t,s) = I + \int_s^t M(\tau_1)\mathrm{d}\tau_1 + \sum_{n=2}^{\infty} \int_s^t M(\tau_1) \int_s^{\tau_1} M(\tau_2) \cdots \int_s^{\tau_{n-1}} M(\tau_n)\mathrm{d}\tau_n \cdots \mathrm{d}\tau_2 \mathrm{d}\tau_1. \tag{8}$$

Moreover, Φ satisfies (see [18] page 26) the following.

Theorem 2.1 *For every $s < t$ in $[c, d]$ the transition matrix Φ satisfies that*

$$\Phi(t, s) = I + \int_s^t M(\sigma)\Phi(\sigma, s)d\sigma$$

and, for every $s \in [c, d]$, the function $t \to \Phi(t, s)$ is piecewise continuously differentiable with derivative

$$\frac{d\Phi(t, s)}{dt} = M(t)\Phi(t, s) \quad \forall t \in [c, d] \setminus E, \tag{9}$$

where $E \subset [c, d]$ is the set at which M fails to be continuous. Moreover, for every $\tau < \alpha < t$ in $[c, d]$, $\Phi(\tau, \tau) = I$, $\Phi(t, \alpha)\Phi(\alpha, \tau) = \Phi(t, \tau)$ and $\Phi(\tau, t) = \Phi^{-1}(t, \tau)$.

The following corollaries are perhaps well known but we could not find a reference. Therefore we give proofs in the Appendix.

Corollary 2.1 *For every $s < t$ in $[c, d]$*

$$\frac{d\Phi(s, t)}{dt} = -\Phi(s, t)M(t) \quad \forall t \in [c, d] \setminus E,$$

with E as in (9).

We denote by $M^\star$ the transpose of a matrix M.

Corollary 2.2 *Let $\hat{M}(t) := -M^\star(t)$ for all $t \in [c, d]$. Then the transition matrix $\hat{\Phi}$ generated by $\hat{M}$ is given by*

$$\hat{\Phi}(t, s) = \Phi^\star(s, t) \quad \forall (t, s) \in [c, d] \times [c, d]. \tag{10}$$

Finally, consider the inhomogeneous linear ODE

$$\dot{x}(t) = M(t)x(t) + b(t) \quad \forall t \in [c, d], x(c) = \beta, \tag{11}$$

where $b : [c, d] \to \mathbb{R}^n$ is a given piecewise continuous function. This equation has the unique solution

$$x(t) = \Phi(t, c)\beta + \int_c^t \Phi(t, \alpha)b(\alpha)\mathrm{d}\alpha. \tag{12}$$

2.2 Back to the finite–horizon case

Definition 2.1 A pair $(\bar{x}, \bar{a}) : [0, T] \to \mathbb{R}^n \times A$ is said to be an optimal solution of the OCP $(4) - (6)$ if $\bar{a}(\cdot)$ maximizes (6) and $\bar{x}(\cdot)$ is the corresponding solution to (4).

We now state our first main result: It gives *necessary conditions* for a pair $(\bar{x}, \bar{a})$ to be an optimal solution to (4)-(6). In particular, $\bar{a}$ depends only on the time parameter, which means that $\bar{a}$ is *independent of the state variable.*

Theorem 2.2 *Under Assumption* 2.1, *if $(\bar{x}, \bar{a}) : [0, T] \to \mathbb{R}^n \times A$ is an optimal solution to the OCP $(4) - (6)$, then $\bar{a}(\cdot)$ depends only on the time parameter t. In fact, for all $t \in [0, T]$, $\bar{a}(t)$ is such that*

$$r_2(t, \bar{a}(t)) + \lambda(t)F_2(t, \bar{a}(t)) = \max_{a \in A} \{r_2(t, a) + \lambda(t)F_2(t, a)\}, \tag{13}$$

where $\lambda^\star(\cdot)$ *satisfies equation* (19) *below. Consequently, the OCP's value function, defined as*

$$V(x_0) := \sup_{a(\cdot)} J(x_0, T, a(\cdot)) \ \ \forall x_0 \in X,$$

is linear in the initial condition x_0*, that is,* $V(x_0) = Mx_0 + N$ *where* M *and* N *are constants depending on* r_1, r_2, F_1, F_2 *and* T*.*

The proof of Theorem 2.2 is given in Section 2.3 below.

Remark 2.1 Recall that (5) requires $r_1 \not\equiv 0$. Otherwise, our result on the linearity of the value function may not hold. See examples 6.6 and 6.7.

The Hamiltonian for the OCP (4)–(6) is given by

$$H(s, x, a, \lambda) = \langle r_1(s), x\rangle + r_2(s, a) + \lambda(F_1(s)x + F_2(s, a)), \quad \lambda \in \mathbb{R}^{1\times n}, \tag{14}$$

where λ in (14) is a **row** $n-$vector.

The following theorem gives *sufficient conditions* for a control $\bar{a}(\cdot)$ satisfying (13) to be optimal.

Theorem 2.3 *Suppose that Assumption* 2.1 *holds. Let* $\bar{a} \in \mathcal{A}$ *and* $\lambda : [0, T] \to \mathbb{R}^{1\times n}$ *satisfy equation* (13) *and equation* (19) *below, and suppose the following:*

(a) A *is a convex set,*
(b) $H(t, \cdot, \cdot, \lambda(t))$ *is concave and continuously differentiable for each* t*, and*
(c) $\bar{a}(t)$ *is an interior point of* A *for each* t*.*

Then $(\bar{x}, \bar{a})$ *is an optimal solution for the OCP* (4) − (6)*.*

Theorem 2.3 follows directly from results on "sufficient conditions of optimality". See, for instance, Theorem 4.10 in [13] or Theorem 2.5 in [29].

2.3 Proof of Theorem 2.2

Consider the value function

$$V(x_0) := \sup_{a(\cdot)} \left\{ \int_0^T [\langle r_1(t), x(t)\rangle + r_2(t, a(t))] \,\mathrm{d}t \right\}. \tag{15}$$

We take the supremum in (15) over all piecewise continuous functions $a(\cdot)$ in $\mathcal{A}[0, T]$.

By the maximum principle [2,13,29], if $(\bar{x}(\cdot), \bar{a}(\cdot))$ is an optimal solution to the OCP (4)–(6) with value function (15), then there exists a function $\lambda : [0, T] \to \mathbb{R}^{1\times n}$ that satisfies the so-called adjoint equation

$$\begin{cases} -\dot{\lambda}(s) = r_1(s) + \lambda(s)F_1(s), \\ \lambda(T) = 0 \end{cases} \tag{16}$$

and, in addition,

$$H(s, \bar{x}(s), \bar{a}(s), \lambda(s)) = \max_{a\in A} H(s, \bar{x}(s), a, \lambda(s)). \tag{17}$$

Note that we can write (16) as

$$\begin{cases} \dot{\lambda}^\star(s) = -F_1^\star(s)\lambda^\star(s) - r_1^\star(s) \\ \lambda^\star(T) = 0, \end{cases} \tag{18}$$

where $M^\star$ denote the transpose of a matrix M.

If Φ is the transition matrix generated by F_1, then Corollary 2.2 implies that (18) has a unique solution given by

$$\lambda^\star(t) = \Phi^\star(0,t)\lambda^\star(0) - \int_0^t \Phi^\star(s,t) r_1^\star(s)\mathrm{d}s \quad \forall t \in [0,T], \tag{19}$$

with

$$\lambda^\star(0) = \Phi^\star(T,0)\left(\int_0^T \Phi^\star(s,T) r_1^\star(s)\mathrm{d}s\right). \tag{20}$$

Observe that we can rewrite $\lambda^\star$ as

$$\lambda^\star(t) = \int_t^T \Phi^\star(s,t) r_1^\star(s)\mathrm{d}s \quad \forall t \in [0,T]. \tag{21}$$

From (14)

$$\max_a H(t,\bar{x}(t),a,\lambda(t)) = h(t,\bar{x}(t)) + \max_a \{r_2(t,a) + \lambda(s)F_2(t,a)\}$$

with $h(t,\bar{x}(t)) := \langle r_1(t), \bar{x}(t)\rangle + \lambda(t)F_1(t)\bar{x}(t)$. Therefore, if $\bar{a}(t) \in A$ satisfies (13), i.e.,

$$r_2(t,\bar{a}(t)) + \lambda(t)F_2(t,\bar{a}(t)) = \max_{a\in A}\{r_2(t,a) + \lambda(t)F_2(t,a)\}, \tag{22}$$

then (17) holds, i.e.,

$$H(t,\bar{x}(t),\bar{a}(t),\lambda(t)) = \max_{a\in A}\{H(t,\bar{x}(t),a,\lambda(t))\}$$

and $\bar{a}(t)$ is a function of $r_2(t,\cdot)$, $F_2(t,\cdot)$ and $\lambda(t)$. This yields the first statement in Theorem 2.2 because, from (21), $\bar{a}(t)$ is a function of the time parameter t only.

To complete the proof of the theorem note that, from (4) and (11)-(12), the state dynamics $\bar{x}(\cdot)$ is given by

$$\bar{x}(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,s)F_2(s,\bar{a}(s))\mathrm{d}s \quad \forall t \in [0,T].$$

whereas, from (15),

$$V(x_0) = \int_0^T \langle r_1(t), \bar{x}(t)\rangle \mathrm{d}t + \int_0^T r_2\left(t,\bar{a}(t)\right)\mathrm{d}t.$$

It follows that for all $x_0 \in \mathbb{R}^n$, $V(x_0) = Mx_0 + N$ where $M = M(F_1, r_1, T)$ and $N = N(F_1, r_1, F_2, r_2, T)$ are given by

$$M := \int_0^T r_1(t)\Phi(t,0)\mathrm{d}t,$$

and

$$N := \int_0^T \left\langle r_1(t), \int_0^t \Phi(t,s)F_2(s,\bar{a}(s))ds\right\rangle \mathrm{d}t + \int_0^T r_2\left(t,\bar{a}(t)\right)\mathrm{d}t,$$

respectively. □

2.4 **The infinite horizon case**

Remark 2.2 As is well known, for infinite-horizon control problems, there are several performance criteria: overtaking optimality, weak overtaking optimality, and so on. (For definitions, see Haurie et al. [12], Carlson et al. [5], Dockner et al. [8]). It turns out that the results for our linear-state control problems are valid for all of these criteria, and the corresponding proofs—based on the maximum principle—are all quite similar. Hence, to avoid being repetitious, we will only consider the discounted case which is the optimality criterion most widely studied in the literature.

We now consider an infinite–horizon OCP with dynamics and running reward given by

$$\dot{x}(t) = F_1(t)x(t) + F_2(t, a(t)), \quad x(0) = x_0, \tag{23}$$

$$r(t, x, a) = e^{-\alpha t} \left(\langle r_1(t), x \rangle + r_2(t, a) \right), \tag{24}$$

with $t \in [0, \infty)$, $r_1 \not\equiv 0$, and state and action spaces $X \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$, respectively.

We consider piecewise–continuous control functions $a(\cdot)$ in the set

$$\mathcal{A}_\infty := \left\{ a : [0, \infty) \to A : \int_0^\infty r(t, x(t), a(t)) \mathrm{d}t < \infty \right\} \tag{25}$$

As in the proof of Theorem 2.2, we wish to use again a maximum principle. In the infinite-horizon case, however, we need to choose a suitable maximum principle. To this end, the following Assumption 2.2 is designed to use Tauchnitz [27] results.

Assumption 2.2 *Let $\varphi(t, x, a)$ be any of the functions r and F in* (23) − (24).

(*a*) *The function φ is uniformly continuous and there is a constant L such that $|\varphi(t, 0, a)| \leq L$ for all (t, a) in $[0, \infty) \times A$.*
(*b*) *For each $t \in [0, \infty)$, there are maximizers of the mappings $a \mapsto r_2(t, a)$ and $a \mapsto F_2(t, a)$.*
(*c*) *α in* (24) *satisfies $0 < L < \alpha$.*

Lemma 2.1 *Under Assumption* 2.2, *the transition matrix Φ generated by F_1 satisfies*

$$|\Phi(t, s)| \leq e^{L(t-s)} \quad \forall \quad 0 < s < t.$$

Proof Note that for all $n \geq 2$

$$\left| \int_s^t F_1(\tau_1) \int_s^{\tau_1} F_1(\tau_2) \cdots \int_s^{\tau_{n-1}} F_1(\tau_n) \mathrm{d}\tau_n \cdots \mathrm{d}\tau_2 \mathrm{d}\tau_1 \right|$$

$$\leq L^n \int_s^t \int_s^{\tau_1} \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n \cdots \mathrm{d}\tau_2 \mathrm{d}\tau_1 = L^n \frac{(t-s)^n}{n!}.$$

Hence, by the Peano-Baker series of Φ (see equation (8)), we get

$$|\Phi(t, s)| \leq \sum_{n=0}^{\infty} L^n \frac{(t-s)^n}{n!} = e^{L(t-s)}$$

for all $0 < s < t$. □

Lemma 2.2 *Suppose that Assumption* 2.2 *holds. Take* $a(\cdot) \in \mathcal{A}_\infty$ *and let* $x(\cdot)$ *be the corresponding trajectory. Then,* $x(\cdot)$ *satisfies*

$$|x(t)| < e^{Lt}(x_0+1) \quad \forall t > 0,$$

and, for all $\rho > 2L$,

$$\int_0^\infty e^{-\rho t}|x(t)|^2\, dt < \infty.$$

Proof From (23) and the variation of parameters formula (12), for all $t > 0$,

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,s)F_2(s,a(s))\mathrm{d}s.$$

Therefore, Lemma 2.1 and Assumption 2.2 imply

$$|x(t)| \le e^{Lt}x_0 + \int_0^t e^{L(t-s)}L\mathrm{d}s = e^{Lt}x_0 + e^{Lt} - 1$$

for all $t > 0$, and the result follows. □

Lemma 2.3 *Suppose that Assumption* 2.2 *holds. Let* $\bar{x}(\cdot)$, $x(\cdot)$ *be trajectories generated by* $a(\cdot) \in \mathcal{A}_\infty$ *with different initial conditions,* $\bar{x}(0) \neq x(0)$*. Then,* $x(\cdot)$ *and* $\bar{x}(\cdot)$ *satisfy for all* $t > 0$

$$|x(t) - \bar{x}(t)| \le e^{Lt}|\bar{x}(0) - x(0)|.$$

Proof From (23) and the variation of parameters formula (12), for all $t > 0$,

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,s)F_2(s,a(s))\mathrm{d}s$$

and

$$\bar{x}(t) = \Phi(t,0)\bar{x}_0 + \int_0^t \Phi(t,s)F_2(s,a(s))\mathrm{d}s.$$

The result follows from Lemma 2.1. □

Under Assumption 2.2, Lemmas 2.2-2.3 imply that Assumptions (**A**1), (**A**2) and (**A**3) in [27] are satisfied by the OCP (23)-(24). Therefore, Theorem 6.1 in [27] holds with $v(t) = e^{-t\rho}$ and $2L < \rho < 2\alpha$, and we obtain the following result.

Theorem 2.4 *Under Assumption* 2.2*, if* $(\bar{x}, \bar{a})$ *is an optimal solution for the OCP* (23)-(24)*, then* $\bar{a}(\cdot)$ *depends only on the time parameter* t*. In fact, for all* $t \in [0,\infty)$*,* $\bar{a}(t)$ *is such that*

$$e^{-\alpha t}r_2(t,\bar{a}(t)) + \lambda(t)F_2(t,\bar{a}(t)) = \max_{a\in A}\left\{e^{-\alpha t}r_2(t,a) + \lambda(t)F_2(t,a)\right\}, \tag{26}$$

where $\lambda(\cdot)$ *satisfies*

$$\lambda^\star(t) = \int_t^\infty \Phi^\star(s,t)r_1^\star(s)\,ds \quad \forall t \in [0,\infty), \tag{27}$$

and Φ *is the transition matrix generated by* F_1*. Consequently, with* J *as in* (6)*, the OCP's value function, defined as*

$$V(x_0) := \sup_{a(\cdot)} J(x_0,\infty,a(\cdot)) \quad \forall x_0 \in X,$$

is linear in the initial condition x_0*, that is,* $V(x_0) = Mx_0 + N$ *where* M *and* N *are constants depending on* r_1, r_2, F_1, F_2.

3 Differential games

Let $N := \{1, \ldots, k\}$, $k \geq 2$, be the set of players. For each player $i \in N$, the set of feasible controls is $A_i \subset \mathbb{R}^{m_i}$. Let the state space be $X = \mathbb{R}^n$ and

$$A := A_1 \times \cdots \times A_k \subset \mathbb{R}^m,$$

with $m := m_1 + \cdots + m_k$.

For each $i \in N$, let $A_{-i} := A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k$. If

$$\bar{a} := (\bar{a}_1, \ldots, \bar{a}_{i-1}, \bar{a}_i, \bar{a}_{i+1}, \ldots, \bar{a}_k)$$

then for all $a_i \in A_i$ we define

$$(a_i, \bar{a}_{-i}) := (\bar{a}_1, \ldots, \bar{a}_{i-1}, a_i, \bar{a}_{i+1}, \ldots, \bar{a}_k).$$

In particular, $\bar{a} = (\bar{a}_i, \bar{a}_{-i})$.

For each player $i \in N$ the strategy space $\mathcal{A}_i$ is the set of piecewise continuous functions $\mathbf{a}_i : [0, T] \to A_i$. Let $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ be the multistrategies space.

Each player $i \in N$ has the objective or payoff function

$$J_i(x_0, \mathbf{a}, T) = \int_0^T \langle r_{1,i}(t), x(t) \rangle + r_{2,i}(t, \mathbf{a}(t))\, \mathrm{d}t, \tag{28}$$

with $\mathbf{a}$ in $\mathcal{A}$ and $r_{1,i} \not\equiv 0$, and subject to the dynamics

$$\dot{x}(t) = F_1(t)x(t) + F_2(t, \mathbf{a}(t)), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{29}$$

For notational ease, and if there no risk of confusion, we will write the solution $x(\cdot; x_0, \mathbf{a})$ of (29) simply as $x(\cdot)$.

Definition 3.1 A Nash equilibrium for the differential game (28) − (29) is a multistrategy $\bar{\mathbf{a}}(\cdot) \in \mathcal{A}$ that satisfies, for each player $i \in N$,

$$J_i(x_0, (\mathbf{a}_i, \bar{\mathbf{a}}_{-i}), T) \leq J_i(x_0, \bar{\mathbf{a}}, T) \quad \forall \mathbf{a}_i \in \mathcal{A}_i,$$

where

$$(\mathbf{a}_i, \bar{\mathbf{a}}_{-i}) := (\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_{i-1}, \mathbf{a}_i, \bar{\mathbf{a}}_{i+1}, \ldots, \bar{\mathbf{a}}_k).$$

We will show below that every Nash equilibrium for the differential game (28)-(29), say $\bar{\mathbf{a}}(\cdot)$, depends only on the time parameter t and, moreover, for every t, $\bar{\mathbf{a}}(t)$ is a Nash equilibrium for the static game (37).

Remark 3.1 To avoid being repetitious, we will only study the finite-horizon case.

3.1 **The finite-horizon case**

The following theorem gives necessary conditions for $\bar{\mathbf{a}}(\cdot)$ to be a Nash equilibrium for (28)-(29).

Theorem 3.1 *Suppose that F_1, F_2 and all $r_{j,i}$ in $(28)-(29)$ satisfy Assumption 2.1. If $\bar{\mathbf{a}}(\cdot)$ is a Nash equilibrium for $(28)-(29)$, then $\bar{\mathbf{a}}(\cdot)$ depends only on the time parameter t and, for every $t \in [0,T]$, $\bar{\mathbf{a}}(t)$ is a Nash equilibrium for the static game (37) below. Consequently, for each player $i \in N$, the corresponding payoff function is linear in the initial condition x_0 :*

$$J_i\left(x_0, \bar{\mathbf{a}}, T\right) = M_i x_0 + N_i \quad \forall x_0 \in X,$$

where M_i and N_i are functions of F_1, F_2, $r_{1,i}$, $r_{2,i}$ and T.

The next theorem states how to obtain a Nash equilibrium for $(28)-(29)$.

Theorem 3.2 *Suppose that F_1, F_2 and all $r_{j,i}$ in $(28)-(29)$ satisfy Assumption 2.1. Let $\bar{\mathbf{a}}(\cdot) \in \mathcal{A}$ be such that, for each $i \in N$, there exists a row vector $\lambda^i : [0,T] \to \mathbb{R}^{1\times n}$ given by (35), and, moreover, for every $t \in [0,T]$, $\bar{\mathbf{a}}(t)$ is a Nash equilibrium for the static game (37). Also, suppose that, for each $i \in N$,*

(a) *A_i is a convex set,*
(b) *$H^i\left(t, \cdot, (\cdot, \bar{\mathbf{a}}_{-i}(t)), \lambda^i(t)\right)$ given by equation (31) is concave in $\mathbb{R}^n \times A_i$ and continuously differentiable for each t, and*
(c) *$\bar{\mathbf{a}}_i(t)$ is in the interior of A_i for each t.*

Then $\bar{\mathbf{a}}(\cdot)$ is a Nash equilibrium for $(28)-(29)$.

Theorem 3.2 follows from Theorem 2.3 and the proof of Theorem 3.1 in Section 3.2.

3.2 **Proof of Theorem 3.1**

Let $\bar{\mathbf{a}} : [0,T] \to \mathbf{A}$ be a Nash equilibrium for (28)-(29) and let $\bar{x}$ be the trajectory generated by $\bar{\mathbf{a}}(\cdot)$. Then for each $i \in N$, $(\bar{x}(\cdot), \bar{\mathbf{a}}_i(\cdot))$ is an optimal solution for an OCP of the form (4)-(6) with

$$\begin{cases} \dot{x}\left(t\right) = F_1(t)x\left(t\right) + F_2\left(t, \left(a_i(t), \bar{\mathbf{a}}_{-i}(t)\right)\right), \\ r_i\left(t, x, a_i\right) = \langle r_{1,i}(t), x\rangle + r_{2,i}\left(t, \left(a_i, \bar{\mathbf{a}}_{-i}(t)\right)\right). \end{cases} \tag{30}$$

We define, for each player i,

$$H^i\left(s, x, (a_i, \bar{\mathbf{a}}_{-i}(s)), \lambda^i\right) := \langle r_{1,i}(s), x\rangle + \lambda^i F_1(t)x + G^i(s, (a_i, \bar{\mathbf{a}}_{-i}(s)), \lambda^i), \tag{31}$$

$$G^i(s, (a_i, \bar{\mathbf{a}}_{-i}(s)), \lambda^i) := r_{2,i}\left(s, \left(a_i, \bar{\mathbf{a}}_{-i}(s)\right)\right) + \lambda^i F_2\left(s, \left(a_i, \bar{\mathbf{a}}_{-i}(s)\right)\right). \tag{32}$$

The maximum principle (see [2,8,12,13,29]) implies the existence of a function $\lambda^i : [0,T] \to \mathbb{R}^{1\times n}$ that satisfies

$$\begin{cases} \dot{\lambda}^i(s) = -H^i_x\left(s, \bar{x}(s), (\bar{\mathbf{a}}_i(s), \bar{\mathbf{a}}_{-i}(s)), \lambda^i(s)\right) \\ \lambda^i(T) = 0. \end{cases} \tag{33}$$

and, for all $s \in [0,T]$,

$$H^i\left(s, \bar{x}(s), (\bar{\mathbf{a}}_i(s), \bar{\mathbf{a}}_{-i}(s)), \lambda^i(s)\right) = \max_{a_i \in A_i} H^i\left(s, \bar{x}(s), (a_i, \bar{\mathbf{a}}_{-i}(s)), \lambda^i(s)\right). \tag{34}$$

From (31)-(32) we can write (33) as

$$\begin{cases} \dot{\lambda}^i(s) = -\lambda^i(s)F_1(s) - r_{1,i}(s), \\ \lambda^i(T) = 0, \end{cases} \tag{35}$$

which means that $\lambda^{i\star}(\cdot)$ satisfies equations (19)-(20) in Section 2.3 with $r_{1,i}$ instead of r_1. Furthermore, from (31)-(32),

$$H^i\left(s, \bar{x}(s), (\bar{\mathbf{a}}_i(s), \bar{\mathbf{a}}_{-i}(s)), \lambda^i(s)\right) = \langle r_{1,i}(s), x\rangle + \lambda^i F_1(s)\bar{x}(s) \\ + G^i(s, (\bar{\mathbf{a}}_i(s), \bar{\mathbf{a}}_{-i}(s)), \lambda^i(s)).$$

Therefore, for all $s \in [0, T]$, $\bar{\mathbf{a}}_i(s)$ satisfies (34) if and only if

$$G^i(s, (\bar{\mathbf{a}}_i(s), \bar{\mathbf{a}}_{-i}(s)), \lambda^i(s)) = \max_{a_i \in A_i} G^i(s, (a_i, \bar{\mathbf{a}}_{-i}(s)), \lambda^i(s)). \tag{36}$$

Since for each player $i \in N$, $\bar{\mathbf{a}}_i(\cdot)$ satisfies (36), then for each $s \in [0, T]$, $\bar{\mathbf{a}}(s)$ is a Nash equilibrium for the static game

$$G(s) := \left(N, \{A_i, i \in N\}, \left\{G^i(s, \cdot, \lambda^i(s)), i \in N\right\}\right). \tag{37}$$

Note that for each $s \in [0, T]$, the static game $G(s)$ depends only on $r_{2,i}(s, \cdot)$, $\lambda^i(s)$, for all $i \in N$, and $F_2(s, \cdot)$. Since each $\lambda^i(\cdot)$ is just a function of time, we obtain that the Nash equilibrium $\bar{\mathbf{a}}(\cdot)$ depends only on the time parameter t. □

Remark 3.2 In general, finding Nash equilibria for a differential game can be a difficult task because we need to deal with coupled OCPs. The so-called Potential Differential Games (PDG)(see [10]) are differential games that can be associated with an OCP in such a way that every solution of the OCP is a Nash equilibrium for the differential game. For instance, consider a finite-horizon differential game with k players, dynamics, and running reward given by:

$$\begin{cases} \dot{x}(t) = F_1(t)x(t) + F_2(t, \mathbf{a}(t)), \\ r_i(t, x, a) = \langle r_1(t), x\rangle + \langle \alpha_i(t), a_i\rangle, \end{cases} \tag{38}$$

where each $\alpha_i : [0, T] \to \mathbb{R}^{m_i}$ is a piecewise continuous function. From Theorem 2 in [10](page 262), (38) is a PDG with associated OCP given by

$$\begin{cases} \dot{x}(t) = F_1(t)x(t) + F_2(t, \mathbf{a}(t)), \\ r(t, x, a) = \langle r_1(t), x\rangle + \sum_{i=1}^{k} \langle \alpha_i(t), a_i\rangle. \end{cases} \tag{39}$$

Therefore, from Theorems 2.2 and 2.3 above, by studying the expression

$$\max_{a \in A}\{\lambda(t)F_2(t, a) + \sum_{i=1}^{k} \langle \alpha_i(t), a_i\rangle\},$$

we can find Nash equilibria for the game (38).

4 Stochastic optimal control problems

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space and let

$$\mathbf{W}(t) = (W_1(t), \ldots, W_d(t)) \quad t \geq 0$$

be a $d-$dimensional $\{\mathcal{F}_t\}_{t\geq 0}$ Brownian motion. We assume that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $\mathbf{W}$, and $\mathcal{F}_0$ includes all $P-$null sets of $\mathcal{F}$.

Consider the following stochastic OCP with dynamics and reward function given by

$$\mathrm{d}x\,(t) = l(t, x(t), a(t))\mathrm{d}t + \sigma(t, x(t))\mathrm{d}\mathbf{W}(t), \quad x(0) = x_0 \in \mathbb{R}^n \tag{40}$$

$$r\,(t, x, a) = \langle r_1(t), x\rangle + r_2\,(t, a) \tag{41}$$

where r_1, r_2 satisfy Assumption 2.1 (with $r_1 \not\equiv 0$) and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $l : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are given by

$$l(t, x, a) = F_1(t)x + F_2\,(t, a)\,, \tag{42}$$

and σ is the $n \times d$ matrix with columns $B_i(t)x + b_i(t)$ $i = 1, \ldots, d$, that is,

$$\sigma(t, x) = (B_1(t)x + b_1(t)| \cdots |B_d(t)x + b_d(t)) \tag{43}$$

Note that (40) can be written as

$$\mathrm{d}x\,(t) = l(t, x(t), a(t))dt + \sum_{i=1}^{d} [B_i(t)x(t) + b_i(t)]\,\mathrm{d}W_i(t).$$

The stochastic OCP is to maximize the objective function

$$J\,(x_0, T, a(\cdot)) = E\left[\int_0^T [\langle r_1(t), x(t)\rangle + r_2\,(t, a(t))]\,\mathrm{d}t\right] \tag{44}$$

over all $\{\mathcal{F}_t\}-$ adapted processes $a : [0, T] \to A$ with piecewise continuous trajectories; we denote this set of controls as $\mathcal{A}$. To ensure that this OCP is well defined we will suppose the following.

Assumption 4.1 *The functions l and σ are measurable and satisfy:*

(a) *Linear growth: For every $T > 0$, there is a constant $K = K(T)$ such that, for all $0 \leq t \leq T$ and $x \in \mathbb{R}^n$,*

$$\|l(t, x, a)\| \leq K\,(1 + \|x\|) \quad \|\sigma(t, x)\| \leq K\,(1 + \|x\|)\,,$$

where $\|\sigma(t, x)\| = (tr\,(\sigma(t, x)\sigma(t, x)^{\star}))^{\frac{1}{2}}$ and $tr(D)$ denotes the trace of a square matrix D.

(b) *Lipschitz conditions: l and σ satisfy (a) in Assumption* 2.1.

The following theorem is a stochastic analogue of Theorem 2.2. That is, it gives conditions under which an optimal control of the problem (40)-(44) does **not** depend on the state process; it only depends on the time parameter.

Theorem 4.1 *Under Assumption* 4.1, *if* $(\bar{x}(\cdot), \bar{a}(\cdot))$ *is an optimal solution to the OCP* (40)−(44), *then* $P-a.s.$ $\bar{a}(\cdot)$ *depends only on the time parameter* t. *In fact, for all* $t \in [0,T]$, $\bar{a}(t)$ *satisfies* $P-a.s.$

$$r_2(t, \bar{a}(t)) + \langle \bar{p}(t), F_2(t, \bar{a}(t)) \rangle = \max_{a \in A} \{ r_2(t,a) + \langle \bar{p}(t), F_2(t,a) \rangle \}.$$

where $\bar{p}(\cdot)$ *satisfies the stochastic equation* (52) *below. Consequently, the OCP's value function, defined as*

$$V(x_0) := \sup_{a(\cdot) \in \mathcal{A}} J(x_0, T, a(\cdot)) \quad \forall x_0 \in X,$$

is linear in the initial condition x_0, *that is,* $V(x_0) = Mx_0 + N$ *where* M *and* N *are constants depending on* l, σ, r_1, r_2 *and* T.

Proof The Hamiltonian for this OCP is given [5, 12, 29] by

$$H(t,x,a,p,q) = \langle p, F_1(t)x + F_2(t,a) \rangle + tr\left[q^\star(\sigma(t,x))\right] + \langle r_1(t), x \rangle + r_2(t,a). \tag{45}$$

Note that the diffusion coefficient σ in equation (40) does not depend on the control process $a(\cdot)$ (see equation (43)). As noted in [29] (see Case 1 in page 119), if $\{(\bar{x}(t), \bar{a}(t))\}_{t \geq 0}$ is a solution for the stochastic OCP (40)−(44), then for almost every $t \in [0,T]$

$$H(t, \bar{x}(t), \bar{a}(t), \bar{p}(t), \bar{q}(t)) = \max_{a \in A} H(t, \bar{x}(t), a, \bar{p}(t), \bar{q}(t)) \quad P-a.s. \tag{46}$$

where $(\bar{p}(\cdot), \bar{q}(\cdot))$ is the unique solution to the following backward stochastic differential equation (SDE) (see Theorem 2.2 in [29] page 349)

$$d\bar{p}^\star(t) = -\left[F_1^\star(t)\bar{p}^\star(t) + \sum_{j=1}^{d} B_j^\star(t)\bar{q}_j(t) + r_1^\star(t) \right] dt + \bar{q}(t) d\mathbf{W}(t), \tag{47}$$

with $\bar{p}(T) = 0$ (here $\bar{p}(t) \in \mathbb{R}^{1 \times n}$ is a **row** vector and $\bar{q}(t) = (\bar{q}_1(t)| \cdots |\bar{q}_d(t)) \in \mathbb{R}^{n \times d}$ where each $\bar{q}_j(t) \in \mathbb{R}^n$ is a **column** vector).

Following the proof of Theorem 2.2 in [29] (page 351) $\bar{p}(\cdot)$ is given by

$$\forall t \in [0,T], \quad \bar{p}^\star(t) = \Delta^{-1}(t)\left(-\int_0^t \Delta(s) r_1^\star(s) ds + E\left[\Theta_T | \mathcal{F}_t\right] \right) \quad P-a.s. \tag{48}$$

where

$$\Theta_T = \int_0^T \Delta(s) r_1^\star(s) ds, \tag{49}$$

and $\Delta(\cdot)$ satisfies the following SDE

$$d\Delta(t) = \Delta(t) F_1^\star(t) dt + \sum_{j=1}^{d} \Delta(t) B_j^\star(t) dW_j(t), \quad \Delta(0) = I. \tag{50}$$

Equations (48)-(50) imply that for all $t \in [0,T]$,

$$\bar{p}^\star(t) = E\left[\int_t^T \Delta^{-1}(t)\Delta(s) r_1^\star(s) ds | \mathcal{F}_t \right]$$

Let $\Psi(\cdot)$ be the solution of the fundamental equation

$$d\Psi(t) = F_1(t)\Psi(t)dt + \sum_{j=1}^{d} B_j(t)\Psi(t)dW_j(t), \quad \Psi(0) = I. \tag{51}$$

Note that for all $t \in [0,T]$, $\Psi(t) = \Delta^\star(t)$ (see equation (50)). This means that we can write $\bar{p}$ as

$$\forall t \in [0,T], \quad \bar{p}(t) = E\left[\int_t^T r_1(s)\Psi(s)\Psi^{-1}(t)ds|\mathcal{F}_t\right] \quad P-a.s. \tag{52}$$

From (51)-(52) for all $t \in [0,T]$, $\bar{p}(t)$ does not depend on the state process $\bar{x}(t)$. Moreover, since

$$\max_{a\in A} H(t,\bar{x}(t),a,\bar{p}(t),\bar{q}(t)) = h(t,\bar{x}(t)) + \max_{a\in A}\{r_2(t,a) + \langle\bar{p}(t),F_2(t,a)\rangle\}$$

with

$$h(t,\bar{x}(t)) := \langle\bar{p}(t),F_1(t)\bar{x}(t)\rangle + tr\left[\bar{q}^\star(t)\sigma(t,\bar{x}(t))\right] + \langle r_1(t),\bar{x}(t)\rangle,$$

it follows that for all $t \in [0,T]$, $\bar{a}(t)$ satisfies $P-a.s.$

$$r_2(t,\bar{a}(t)) + \langle\bar{p}(t),F_2(t,\bar{a}(t))\rangle = \max_{a\in A}\{r_2(t,a) + \langle\bar{p}(t),F_2(t,a)\rangle\}. \tag{53}$$

Thus $\bar{a}(t)$ depends only on t.

Finally, to prove the linearity of $V(\cdot)$, note that from Theorem 8.5.2 in [1](page 141), $\bar{x} : [0,T]\times\Omega \to \mathbb{R}^n$ can be written as

$$\bar{x}(t) = \Psi(t)x_0 + \Psi(t)\int_0^t \Psi^{-1}(s)d\bar{Y}(s) \tag{54}$$

where

$$d\bar{Y}(t) = \left(F_2(t,\bar{a}(t)) - \sum_{j=1}^{d} B_j(t)b_j(t)\right)dt + \sum_{j=1}^{d} b_j(t)dW_j(t). \tag{55}$$

Since

$$V(x_0) = E\left[\int_0^T [\langle r_1(t),\bar{x}(t)\rangle + r_2(t,\bar{a}(t))]\mathrm{d}t\right],$$

it follows that

$$V(x_0) = Mx_0 + N$$

with

$$M = M(F_1,r_1,B_1,\ldots,B_d,T) := E\left[\int_0^T r_1(t)\Psi(t)\mathrm{d}t\right]$$

and

$$\begin{aligned} N &= N(F_1,r_1,F_2,r_2,\sigma,T) \\ &:= E\left[\int_0^T \left\langle r_1(t),\Psi(t)\int_0^t \Psi^{-1}(s)\mathrm{d}\bar{Y}(s)\right\rangle\mathrm{d}t + \int_0^T r_2(t,\bar{a}(t))\mathrm{d}t\right]. \end{aligned}$$

The proof is complete. □

The following Theorem 4.2 is Theorem 5.2 in [29] (page 138) restated for the OCP (40)-(41). As in the proof of Theorem 4.1, it is important to note that the diffusion coefficient in equation (40) does not depend on the control process $a(\cdot)$ (see equation (43)). Therefore, if (a) and (b) in Theorem 4.2 below hold, then it suffices that $\bar{a} \in \mathcal{A}$ satisfies equation (53) for it to be an optimal solution of (40)-(41) (see Case 1 in [29], page 119).

Theorem 4.2 *Suppose that Assumption* 4.1 *holds. Let* $(\bar{p}(\cdot), \bar{q}(\cdot))$ *be a solution of equation* (47), *and suppose the following:*

(a) *A is a convex set,*
(b) *$H(t, \cdot, \cdot, \bar{p}(t), \bar{q}(t))$ given by* (45) *is concave for all* $t \in [0, T]$.

If $\bar{a}(\cdot) \in \mathcal{A}$ *satisfies* (53) $P - a.s.$, *then* $\bar{a}(\cdot)$ *is an optimal control for the OCP* (40) $-$ (44), *and* $\bar{a}(\cdot)$ *depends only on the time parameter* t.

4.1 **The infinite-horizon case**

In the finite-horizon case, we showed that the adjoint equations and the maximum condition of the maximum principle can be expressed as in (47) and (53), respectively. This immediately implies that the optimal control is a function of the time parameter only.

For the infinite-horizon case, there exist a maximum principle with similar adjoint equations and maximum condition to those in the finite-horizon case (see equations (3),(4), (17) and Theorem 4.1 in [11]). In fact, due to the class of problems we are considering (there are no jumps in the dynamics of the system), equations (4) and (17) in [11] can be written as equations (47) and (53), respectively; that is

$$\mathrm{d}\bar{p}^{\star}(t) = -\left[F_1^{\star}(t)\bar{p}(t) + \sum_{j=1}^{d} B_j^{\star}(t)\bar{q}_j(t) + r_1^{\star}(t) \right] \mathrm{d}t + \bar{q}(t)\mathrm{d}\mathbf{W}(t), \tag{56}$$

for all $t \in [0, \infty]$, and

$$r_2(t, \bar{a}(t)) + \langle \bar{p}(t), F_2(t, \bar{a}(t)) \rangle = \max_{a \in A} \{ r_2(t, a) + \langle \bar{p}(t), F_2(t, a) \rangle \}.$$

Definition 4.1 Given $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, we denote by $M_{\mathcal{F}_t}^{2;\alpha}([0, \infty); \mathbb{R}^k)$ the space of all $\mathcal{F}_t-$progressively measurable processes $z(\cdot)$ with values in $\mathbb{R}^k$ such that

$$E \left[\int_0^{\infty} |z(t)|^2 e^{\alpha t} \mathrm{d}t \right] < \infty.$$

To verify the existence and uniqueness of the solution of equation (56) we need the following assumption

Assumption 4.2 (a) *There exists* $C_0 > 0$ *such that for all* $t \geq 0$, $|F_1(t)| \leq C_0$.
(b) *There exists* $C > 0$ *such that*

$$|\sum_{j=1}^{d} B_j(t)(q_j^1 - q_j^2)| \leq C||q^1 - q^2|| \quad \forall t > 0, \quad \forall q^1, q^2 \in \mathbb{R}^{n \times d},$$

where $||q|| = (tr(qq^{\star}))^{\frac{1}{2}}$.
(c) $r_1 \in M_{\mathcal{F}_t}^{2;\kappa}([0, \infty); \mathbb{R}^n)$ $(r_1 \not\equiv 0)$ *where* κ *satisfies*

$$\kappa > 2C_0 + 2C^2 + \delta$$

for some $\delta > 0$.

Remark 4.1 Assumption 4.2 is the standard hypothesis that we need in order to obtain the existence and uniqueness of the SDE (56); see for instance hypothesis H4 in [23] or Section 3 in [11].

Lemma 4.1 *Under Assumption* 4.2*, the backward SDE (56) has a unique solution* $(\bar{p}(\cdot), \bar{q}(\cdot))$ *in* $M^{2;\kappa}_{\mathcal{F}_t}([0,\infty); \mathbb{R}^n \times \mathbb{R}^{n\times d})$*. Moreover,*

$$\lim_{T\to\infty} E\left[\int_0^\infty |\bar{p}(t) - \bar{p}_T(t)|^2 e^{\kappa t}\, dt\right] = 0$$

where

$$\bar{p}_T(t) = \begin{cases} E\left[\int_t^T r_1(s)\Psi(s)\Psi^{-1}(t)ds|\mathcal{F}_t\right] & t\in[0,T], \\ 0 & t > T, \end{cases}$$

and $\Psi(\cdot)$ *is the solution of the fundamental equation* (51)*.*

Proof The existence and uniqueness of a solution of (56) follows from Theorem 4 in [23] because Assumption 4.2 implies hypothesis H4 in [23] (just take G in H4 as $G(t,p,q) = F_1^\star(t)p + \sum_{j=1}^d B_j^\star(t)q_j$, see page 79 in [23]).

Furthermore, from the proof of Theorem 4 in [23],

$$\lim_{T\to\infty} (\bar{p}_T(\cdot), \bar{q}_T(\cdot)) = (\bar{p}(\cdot), \bar{q}(\cdot))$$

where convergence is understood in the sense of $M^{2;\kappa}_{\mathcal{F}_t}([0,\infty); \mathbb{R}^n \times \mathbb{R}^{n\times d})$, and for each $T > 0$, $(\bar{p}_T(\cdot), \bar{q}_T(\cdot))$ is the unique solution of the infinite-horizon backward SDE

$$\mathrm{d}(I_{[0,T]}(t)\bar{p}^\star(t)) = -\left[I_{[0,T]}(t)\left(F_1^\star(t)\bar{p}^\star(t) + \sum_{j=1}^d B_j^\star(t)\bar{q}_j(t) + r_1^\star(t)\right)\right]\mathrm{d}t$$
$$+(I_{[0,T]}(t)\bar{q}(t))\mathrm{d}\mathbf{W}(t), \quad t \geq 0.$$

Here $I_{[0,T]}(\cdot)$ is the characteristic (or indicator) function of the time interval $[0,T]$.

It follows that $\bar{p}_T(\cdot)$ coincides, in the time interval $[0,T]$, with the solution of equation (47) (see equation (48)). □

We consider piecewise–continuous and $\mathcal{F}_t$–progressively measurable control processes $a(\cdot)$ in the set

$$\mathcal{A}_\infty := \left\{a : [0,\infty) \to A : E\left[\int_0^\infty r(t, x(t), a(t))\mathrm{d}t\right] < \infty\right\}. \tag{57}$$

Lemma 4.2 *Given* $\bar{a}, a \in \mathcal{A}_\infty$*, let* $\bar{x}$*,* x *be, respectively, the corresponding dynamics given by equation (40)(with* $T = \infty$*). Then*

$$\bar{x}(t) - x(t) = \int_0^t \Psi(t)\Psi^{-1}(s)(F_2(s,\bar{a}(s)) - F_2(s,a(s)))ds \quad \forall t > 0$$

where $\Psi(t)$ *is the solution of the fundamental equation (51).*

Proof See equations (54)-(55). □

Assumption 4.3 *Given* $\bar{a}, a \in \mathcal{A}_\infty$*, let* $\bar{x}$*,* x *be as in Lemma* 4.2*. Let* $(\bar{p}(\cdot), \bar{q}(\cdot))$ *be the solution of the backward SDE* (56)*. The following holds:*

(*i*) *For all* $T > 0$,

$$E\left[\int_0^T \left(\int_0^t \Psi(t)\Psi^{-1}(s)(F_2(s,\bar{a}(s)) - F_2(s,a(s))\,ds\right)^{\star} \times (\bar{q}(t)\bar{q}^{\star}(t))\right.$$
$$\left. \times \left(\int_0^t \Psi(t)\Psi^{-1}(s)(F_2(s,\bar{a}(s)) - F_2(s,a(s)))\,ds\right) dt\right] < \infty$$

and

$$E\left[\int_0^T \bar{p}(t)\left(\sigma(t,x(t))\sigma(t,x(t))^{\star}\right) p^{\star}(t)\,dt\right] < \infty,$$

where σ *is given by equation* (43).

(*ii*) $F_2(t,a)$ *and* $r_2(t,a)$ *are differentiable in* a, *and*

$$E\left[\left|\frac{\partial}{\partial a}F_2(t,\bar{a}(t)) + \frac{\partial}{\partial a}r_2(t,\bar{a}(t))\right|^2\right] < \infty;$$

(*iii*) *The Hamiltonian* H *in equation* (45) *satisfies*

$$E\left[\int_0^\infty |H(t,\bar{x}(t),\bar{a}(t),\bar{p}(t),\bar{q}(t)|\,dt\right] < \infty.$$

It follows from Theorem 4.1 in [11] and Lemmas 4.1 and 4.2 that Theorem 4.2 above has the following analogue for the infinite-horizon case.

Theorem 4.3 *Suppose that the hypotheses in Theorem* 4.2 *hold in the time interval* $[0,\infty)$. *Also, assume that the Assumptions 4.2 and 4.3 hold. If* $\bar{a}(\cdot) \in \mathcal{A}_\infty$ *satisfies for all* $t > 0$

$$r_2(t,\bar{a}(t)) + \langle \bar{p}(t), F_2(t,\bar{a}(t))\rangle = \max_{a\in A}\{r_2(t,a) + \langle \bar{p}(t), F_2(t,a)\rangle\} \quad P-a.s.$$

and the transversality condition

$$0 \le E\left[\limsup_{t\to\infty} \quad \bar{p}(t) \times \left(\int_0^t \Psi(t)\Psi^{-1}(s) \times (F_2(s,a(s))ds - F_2(s,\bar{a}(s))\,ds\right)\right] < \infty,$$

then $\bar{a}(\cdot)$ *is an optimal control for the OCP* (40)$-$(41), *and* $\bar{a}(\cdot)$ *depends on the time parameter* t *only.*

4.2 **The Certainty Equivalence Principle**

The following paragraph, borrowed from [14], summarizes Theil's [28] idea behind the certainty equivalence principle.

" According to Theil [28], certainty equivalence means that a decision agent who maximizes expected utility and takes actions based on the information available at the time of taking the decision, may neglect the disturbances and to suppose that the uncertain elements are settled at their mean values."

In this section we show how Theil's idea holds for the stochastic OCP (40)-(44). (Similar arguments hold for infinite-horizon problems and stochastic games). To this end, recall that the optimal control $\bar{a}(\cdot)$ satisfies, for all $t \in [0,T]$, the maximality condition

$$r_2(t,\bar{a}(t)) + \langle \bar{p}(t), F_2(t,\bar{a}(t)) = \max_{a\in A}\{r_2(t,a) + \langle \bar{p}(t), F_2(t,a)\rangle\} \quad P-a.s.$$

where $\bar{p}(\cdot)$ is given by

$$\forall t \in [0,T], \quad \bar{p}(t) = E\left[\int_t^T r_1(s)\Psi(s)\Psi^{-1}(t)ds|\mathcal{F}_t\right] \quad P-a.s.$$

By Remark 4.5 in [4] (see page 96), $\bar{p}(\cdot)$ can be written as

$$\forall t \in [0,T], \quad \bar{p}(t) = \int_t^T r_1(s)E\left[\Psi(s)\Psi^{-1}(t)|\mathcal{F}_t\right] ds \quad P-a.s.$$

The following proposition relates Ψ and Φ, where Φ is the transition matrix in Section 2.3.

Proposition 4.1 *Let $0 \le t \le s \le T$ and let Z be $\mathcal{F}_t-$measurable and bounded random variable in $\mathbb{R}^n$. Then*

$$E\left[\Psi(s)\Psi^{-1}(t)Z\right] = \Phi(s,t)E[Z], \tag{58}$$

where $s \to \Phi(s,t)$ is the unique solution of the linear ODE

$$\forall s \in [t,T], \quad \frac{d}{ds}\Phi(s,t) = F_1(s)\Phi(s,t), \quad \Phi(t,t) = I.$$

Proof. By (51),

$$\Psi(s) = I + \int_0^s F_1(\tau)\Psi(\tau)d\tau + \sum_{j=1}^d \int_0^s B_j(\tau)\Psi(\tau)dW_j(\tau)$$

and so

$$\Psi(s) = \Psi(t) + \int_t^s F_1(\tau)\Psi(\tau)d\tau + \sum_{j=1}^d \int_t^s B_j(\tau)\Psi(\tau)dW_j(\tau).$$

Since $\Psi^{-1}(t)$ and Z are $\mathcal{F}_t-$measurable (see [4] page 204) we obtain

$$\Psi(s)\Psi^{-1}(t)Z = Z + \int_t^s F_1(\tau)\Psi(\tau)\Psi^{-1}(t)Zd\tau + \sum_{j=1}^d \int_t^s B_j(\tau)\Psi(\tau)\Psi^{-1}(t)ZdW_j(\tau).$$

Moreover, since Z is bounded, Remark 9.2 in [4](see page 264) implies

$$E\left[\Psi(s)\Psi^{-1}(t)Z\right] = E\left[Z\right] + \int_t^s F_1(\tau)E\left[\Psi(\tau)\Psi^{-1}(t)Z\right] d\tau$$

The result follows from the last equation. □

Corollary 4.1

$$E\left[\Psi(s)\Psi^{-1}(t)|\mathcal{F}_t\right] = \Phi(s,t) \quad \forall 0 \le t \le s \le T, \tag{59}$$

and

$$\bar{p}(t) = \lambda(t) \quad \forall t \in [0,T], \tag{60}$$

where $\lambda(\cdot)$ is the costate function in the deterministic case (see Section 2).

To conclude, from (58)-(60) and Theorem 4.1 we obtain the **certainty equivalence principle** in the following Theorem 4.4, because the *stochastic process* $\bar{p}(\cdot)$ in Theorem 4.1 is replaced by the *deterministic function* $\lambda(\cdot)$ in (19)-(21).

Theorem 4.4 *Under Assumption* 4.1, *if* $(\bar{x}(\cdot), \bar{a}(\cdot))$ *is an optimal solution to the OCP* (40)−(44), *then $P-a.s.$ $\bar{a}(\cdot)$ depends only on the time parameter t. In fact, for all $t \in [0,T]$, $\bar{a}(t)$ satisfies $P-a.s.$*

$$r_2(t,\bar{a}(t)) + \langle\lambda(t), F_2(t,\bar{a}(t))\rangle = \max_{a\in A}\{r_2(t,a) + \langle\lambda(t), F_2(t,a)\rangle\}.$$

where $\lambda(\cdot)$ is as in Section 2.

5 Stochastic differential games

Let $N := \{1, \ldots, k\}$, $k \geq 2$, $A_i \subset \mathbb{R}^{m_i}$, $X = \mathbb{R}^n$ and

$$A := A_1 \times \cdots \times A_k \subset \mathbb{R}^m$$

with $m := m_1 + \cdots + m_k$, be as in Section 3. Moreover, we again write, for each $i \in N$, $A_{-i} := A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k$.

In addition, as in Section 4, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space and $\{\mathbf{W}(t)$ a $d-$dimensional $\{\mathcal{F}_t\}_{t\geq 0}$ Brownian motion.

For each player $i \in N$, the strategy space $\mathcal{A}_i$ is the set of $\{\mathcal{F}_t\}_{t\geq 0}-$adapted processes $\mathbf{a}_i : [0, T] \times \Omega \to A_i$ with piecewise-continuous trajectories. Let $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ be the multistrategies space.

For each player $i \in N$, the objective function is

$$J_i\left(x_0, \mathbf{a}, T\right) = E\left[\int_0^T \langle r_{1,i}(t), x(t)\rangle + r_{2,i}\left(t, \mathbf{a}(t)\right) \mathrm{d}t\right] \tag{61}$$

for $\mathbf{a} \in \mathcal{A}$, $r_{1,i} \not\equiv 0$, and subject to the dynamics

$$\mathrm{d}x\,(t) = l(t, x(t), \mathbf{a}(t))\mathrm{d}t + \sigma(t, x(t))\mathrm{d}\mathbf{W}(t), \quad \textit{with} \quad x(0) = x_0 \in \mathbb{R}^n, \tag{62}$$

with coefficients

$$l(t, x, \mathbf{a}) = F_1(t)x + F_2\left(t, \mathbf{a}\right) \tag{63}$$

and

$$\sigma(t, x) = \left(B_1(t)x + b_1(t)| \cdots |B_d(t)x + b_d(t)\right). \tag{64}$$

Definition 5.1 A Nash equilibrium for the stochastic differential game (61)-(62) is a multistrategy $\bar{\mathbf{a}}(\cdot) \in \mathcal{A}$ that satisfies for each player $i \in N$

$$J_i\left(x_0, (\mathbf{a}_i, \bar{\mathbf{a}}_{-i}), T\right) \leq J_i\left(x_0, \bar{\mathbf{a}}, T\right) \quad \forall \mathbf{a}_i \in \mathcal{A}_i,$$

where

$$(\mathbf{a}_i, \bar{\mathbf{a}}_{-i}) := \left(\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_{i-1}, \mathbf{a}_i, \bar{\mathbf{a}}_{i+1}, \ldots, \bar{\mathbf{a}}_k\right).$$

Remark 5.1 As in Section 3, we will only study the finite-horizon case.

5.1 The finite-horizon case

Due to the similarities between Theorems 2.2-2.3 and Theorems 4.1-4.2 (for deterministic and stochastic OCPs, respectively), we can expect that Theorems 3.1-3.2 will have their corresponding stochastic analogues. Indeed, in this case we have the following.

Theorem 5.1 *Suppose that l and σ satisfy Assumption* 4.1, *and all $r_{j,i}$ in* (61) *satisfy Assumption* 2.1. *If $\bar{\mathbf{a}}(\cdot)$ is a Nash equilibrium for* $(61) - (62)$, *then $\bar{\mathbf{a}}(\cdot)$ depends only on the time parameter t and, for every $t \in [0, T]$, $\bar{\mathbf{a}}(t)$ is a Nash equilibrium for the static game* (72) *below. Consequently, for each player $i \in N$*

$$J_i\left(x_0, \bar{\mathbf{a}}, T\right) = M_i x_0 + N_i$$

where M_i and N_i are functions of l, σ, $r_{1,i}$, $r_{2,i}$ and T.

We now have the analog of Theorem 3.2 on how to obtain a Nash equilibrium for (61)−(62).

Theorem 5.2 *Suppose that l and σ satisfy Assumption* 4.1, *and all $r_{j,i}$ in* (61) *satisfy Assumption* 2.1. *Let $\bar{\mathbf{a}}(\cdot) \in \mathcal{A}$ be such that there exists, for each $i \in N$, $\bar{p}^i(\cdot)$ given by* (70), *and, futhermore, for each $t \in [0,T]$, $\bar{\mathbf{a}}(t)$ is a Nash equilibrium for the static game* (72) *below. Moreover, suppose that, for each $i \in N$,*

(a) A_i *is a convex set,*
(b) $H^i\left(t, \cdot, (\cdot, \bar{\mathbf{a}}_{-i}(t)), \bar{p}^i(t), \bar{q}^i(t)\right)$ *in equation (66) is concave in $\mathbb{R}^n \times A_i$ for all $t \in [0,T]$.*

Then $\bar{\mathbf{a}}(\cdot)$ is a Nash equilibrium for (61)−(62).

Theorem 5.2 follows from Theorem 4.2 and the proof of Theorem 5.1.

5.2 **Proof of Theorem 5.1**

Let $\bar{\mathbf{a}} \in \mathcal{A}$ be a Nash Equilibrium for (61)-(62) and let $\bar{x}$ be the state process generated by $\bar{\mathbf{a}}(\cdot)$. Then for each $i \in N$, $(\bar{x}(\cdot), \bar{\mathbf{a}}_i(\cdot))$ is an optimal solution for the stochastic OCP with state dynamics and running payoff

$$\begin{cases} \mathrm{d}x(t) = l(t, x(t), (\bar{\mathbf{a}}_i(t), \bar{\mathbf{a}}_{-i}(t)))\mathrm{d}t + \sigma(t, x(t))\mathrm{d}\mathbf{W}(t), \\ r_i(t, x, a_i) = \langle r_{1,i}(t), x\rangle + r_{2,i}(t, (a_i, \bar{\mathbf{a}}_{-i}(t))), \end{cases} \tag{65}$$

respectively. Consider the Hamiltonian for player i given by

$$\begin{aligned} H^i(t, x, (a_i, a_{-i}), p, q) := \langle r_{1,i}(t), x\rangle + \langle p, F_1(t)x\rangle + tr\left[q^\star(\sigma(t,x))\right] \\ + G_H^i(t, (a_i, a_{-i}), p) \end{aligned} \tag{66}$$

with

$$G_H^i(t, (a_i, a_{-i}), p) := r_{2,i}(t, (a_i, a_{-i})) + \langle p, F_2(t, (a_i, a_{-i}))\rangle. \tag{67}$$

The maximum principle (see [29] page 118) implies the existence of an $\{\mathcal{F}_t\}-$ adapted process $(\bar{p}^i(\cdot), \bar{q}^i(\cdot))$ which is the unique solution for the following backward SDE (see Theorem 2.2 in [29] page 349)

$$\mathrm{d}\bar{p}^{i\star}(t) = -\left[F_1^\star(t)\bar{p}^{i\star}(t) + \sum_{j=1}^{d} B_j^\star(t)\bar{q}_j^i(t) + r_{1,i}^\star(t)\right]\mathrm{d}t + \bar{q}^i(t)\mathrm{d}\mathbf{W}(t), \tag{68}$$

with $\bar{p}^i(T) = 0$, and for all $t \in [0,T]$,

$$H^i\left(t, \bar{x}(t), (\bar{\mathbf{a}}_i(t), \bar{\mathbf{a}}_{-i}(t)), \bar{p}^i(t), \bar{q}^i(t)\right) = \max_{a_i \in A_i} H^i\left(t, \bar{x}(t), (a_i, \bar{\mathbf{a}}_{-i}(t)), \bar{p}^i(t), \bar{q}^i(t)\right). \tag{69}$$

Following the proof of Theorem 2.2 in [29] (page 351), $\bar{p}^i(\cdot)$ is given by

$$\forall t \in [0,T], \quad \bar{p}^i(t) = E\left[\int_t^T r_{1,i}(s)\Psi(s)\Psi^{-1}(t)ds|\mathcal{F}_t\right] \quad P-a.s., \tag{70}$$

where $\Psi(\cdot)$ is the solution of the fundamental equation (51).

From (66)-(69), for all $t \in [0,T]$, $\bar{\mathbf{a}}_i(t)$ satisfies (69) if and only if $\bar{\mathbf{a}}_i(t)$ maximizes (67), i.e.,

$$G_H^i(t, (\bar{\mathbf{a}}_i(t), \bar{\mathbf{a}}_{-i}(s)), \bar{p}^i(t)) = \max_{a_i \in A_i} G_H^i(t, (a_i, \bar{\mathbf{a}}_{-i}(t)), \bar{p}^i(t)). \tag{71}$$

Since for each player $i \in N$, $\bar{\mathbf{a}}_i(\cdot)$ satisfies (71), then for each $t \in [0,T]$, $\bar{\mathbf{a}}(t)$ is a Nash equilibrium for the static game

$$G_H(t) := \left(N, \{A_i, i \in N\}, \left\{G_H^i(t, \cdot, \bar{p}^i(t)), i \in N\right\}\right). \tag{72}$$

Note that, by (67), for each $t \in [0,T]$, the static game $G(t)$ depends only on $r_{2,i}(t,\cdot)$ and $\bar{p}^i(t)$ for all $i \in N$, and $F_2(t,\cdot)$. Futhermore, from (70), each $\bar{p}^i(\cdot)$ does not depend on $\bar{x}(\cdot)$, so the Nash equilibrium $\bar{\mathbf{a}}(\cdot)$ depends only on the time parameter t. □

Remark 5.2. Note that this kind of game will satisfy the Certain Equivalence Principle because each $\bar{p}^i(\cdot)$ is given by equation (70) (see Proposition 4.1 and Corollary 4.1). Moreover,

$$\forall i \in \{1, \ldots, k\}, \quad \bar{p}^i(t) = \lambda^i(t) \quad \forall t \in [0,T],$$

where $\lambda^i(\cdot)$ is the costate vector of player i in the corresponding deterministic game (see Section 3).

6 Examples

In this section, first, we comment on some published examples that illustrate our results. All of these examples use the maximum principle, as in the previous sections.

Finally, we present some examples based on the so-called "verification theorems" in *dynamic programming.* Namely, first, we obtain a sufficiently smooth solution to the associated Hamilton-Jacobi-Bellman (HJB) equation, which gives us the OCP's value function; this function is then used to obtain an optimal control. In our present situation, the first step is trivial because we already know that the value function is linear in the state variable, i.e.,

$$V(t,x) = Mx + N \tag{73}$$

for some constants M and N. Hence we only need to determine these constants and then we find an optimal control.

6.1 **Some already known examples**

General classes of linear-state differential games (even more general than (3.1)-(3.2) above) are introduced in Dockner et al. [8], Section 7.2 and also in [12], page 262. Particular examples are also introduced in, for instance, Section 7.12.2 in Haurie et al. [12]. Similarly, Bacchiega et al. [3] and Jorgensen et al. [16] study important properties and particular applications of linear-state differential games. Other applications can be seen in Long [19], [20].

Example 6.1 In general, calculating the costate function $\lambda(\cdot)$ in (19)-(20), might be difficult because the transition matrix Φ, generated by F_1, is not easy to calculate. A stronger hypothesis needs to be made in order to get a simple expression for Φ: for example, if for all $s,t \in [0,T]$, we have $F_1(s)F_1(t) = F_1(t)F_1(s)$ (as in the time-invariant case) then

$$\Phi(t,s) = \exp\left(\int_s^t F_1(\tau)\mathrm{d}\tau\right). \tag{74}$$

See [18] page 26 for a proof of (74).

Consider the OCP with $X = \mathbb{R}$ and $A = \mathbb{R}^m$ and

$$\begin{cases} \dot{x}(t) = F_1(t)x(t) + F_2(t, a(t)) \\ r(t, x, a) = r_1(t)x + a^\star Q(t)a \end{cases} \tag{75}$$

where $F_1, r_1 : [0, T] \to \mathbb{R}$ are piecewise continuous functions ($r_1 \not\equiv 0$) and $F_2 : [0, T] \times A \to \mathbb{R}$, $Q : [0, T] \to \mathbb{R}^{m \times m}$ are such that for all $0 \le t \le T$, $Q(t)$ is strictly negative definite, and $F_2(t, \cdot)$ is a concave and continuously differentiable.

From (19), (20), (21) and (74) the costate function $\lambda(\cdot)$ is given by

$$\lambda(t) = \exp\left(-\int_0^t F_1(s)\mathrm{d}s\right)\lambda(0) - \int_0^t \exp\left(-\int_s^t F_1(\tau)\mathrm{d}\tau\right) r_1(s)\mathrm{d}s \tag{76}$$

with

$$\lambda(0) = \exp\left(\int_0^T F_1(s)\mathrm{d}s\right)\int_0^T \exp\left(-\int_s^T F_1(\tau)\mathrm{d}\tau\right) r_1(s)\mathrm{d}s, \tag{77}$$

which implies

$$\lambda(t) = \int_t^T \exp\left(\int_t^s F_1(\tau)\mathrm{d}\tau\right) r_1(s)\mathrm{d}s \quad \forall t \in [0, T]. \tag{78}$$

If $\bar{a}(\cdot)$ is an optimal control for (75), then for all $t \in [0, T]$,

$$\bar{a}(t)^\star Q(t)\bar{a}(t) + \lambda(t)F_2(t, \bar{a}(t) = \max_{a \in A}\{a^\star Q(t)a + \lambda(t)F_2(t, a)\} \tag{79}$$

Note that $a \to a^\star Q(t)a + \lambda(t)F_2(t, a)$ is continuously differentiable. This means that for all $t \in [0, T]$, $\bar{a}(t)$ satisfies

$$\frac{\partial}{\partial a}\left(a^\star Q(t)a + \lambda(t)F_2(t, a)\right)|_{a=\bar{a}(t)} = 0 \tag{80}$$

where

$$\frac{\partial}{\partial a}\left(a^\star Q(t)a + \lambda(t)F_2(t, a)\right) = a^\star\left(Q(t) + Q(t)^\star\right) + \lambda(t)\frac{\partial}{\partial a}F_2(t, a). \tag{81}$$

Therefore, if $\bar{a}(\cdot)$ is an optimal control for (75), then for all $t \in [0, T]$,

$$\bar{a}(t) = -\lambda(t)\left(Q(t) + Q(t)^\star\right)^{-1}\left(\frac{\partial}{\partial a}F_2(t, a)|_{a=\bar{a}(t)}\right)^\star, \tag{82}$$

where $Q(t)+Q(t)^\star$ is invertible because it is a symmetric and strictly negative definite matrix.

Now suppose that F_2 satisfies one of the following cases

1. $F_2(t, a) = D(t)a$ with $D : [0, T] \to \mathbb{R}^{1 \times m}$ a piecewise continuous function,
2. $F_2(t, a) = a^\star D(t)a$ with $D(t)$ a positive negative $m \times m$- matrix.

If F_2 satisfies 1, then from (82)

$$\bar{a}(t) = -\lambda(t)\left(Q(t) + Q(t)^\star\right)^{-1}\left(D(t)\right)^\star \quad \forall t \in [0, T].$$

If F_2 satisfies 2, then from (80) and (81)

$$\left[(Q(t) + \lambda(t)D(t)) + (Q(t) + \lambda(t)D(t))^\star\right]\bar{a}(t) = 0 \quad \forall t \in [0, T]. \tag{83}$$

Note that $(Q(t) + \lambda(t)D(t))$ is a strictly negative definite matrix if for all $t \in [0,T]$, $\lambda(t) \geq 0$. From (76) and (77), $\lambda(t) \geq 0$ if and only if

$$\lambda(0) \geq \int_0^t \exp\left(\int_0^s F_1(\sigma)\mathrm{d}\sigma\right) r_1(s)\mathrm{d}s.$$

Since we can writte (77) as

$$\lambda(0) = \int_0^T \exp\left(\int_0^s F_1(\sigma)\mathrm{d}\sigma\right) r_1(s)\mathrm{d}s,$$

then it suffices that $r_1 \geq 0$ in order that, for all $t \in [0,T]$, $\lambda(t) \geq 0$.

It follows that if $r_1 \geq 0$, then for all $t \in [0,T]$, $(Q(t) + \lambda(t)D(t))$ is a strictly negative definite matrix which implies that for all $t \in [0,T]$

$$(Q(t) + \lambda(t)D(t)) + (Q(t) + \lambda(t)D(t))^\star$$

is invertible. Therefore, it follows from (83) that for all $t \in [0,T]$, $\bar{a}(t) = 0$. ♢

Example 6.2 Consider a finite-horizon differential game with

$$\begin{cases} \dot{\mathbf{x}}(t) = F_1(t)\mathbf{x}(t) + F_2(t, \mathbf{a}(t)), \\ r_i(t, \mathbf{x}, \mathbf{a}) = r_{1,i}(t)\mathbf{x} + \mathbf{a}^\star Q_i(t)\mathbf{a}, \end{cases} \tag{84}$$

where for all $i \in N$, the functions $F_1, r_{1,i} : [0,T] \to \mathbb{R}$ are piecewise continuous functions ($r_{1,i} \not\equiv 0$) and $F_2 : [0,T] \times A \to \mathbb{R}$, $Q_i : [0,T] \to \mathbb{R}^{k\times k}$ are such that for all $0 \leq t \leq T$, $Q_i(t)$ is strictly negative definite and $F_2(t,\cdot)$ is concave and continuously differentiable.

From (35), for each player i the costate function $\lambda^i(\cdot)$ is given by

$$\lambda^i(t) = \exp\left(-\int_0^t F_1(s)ds\right)\lambda(0) - \int_0^t \exp\left(-\int_s^t F_1(\sigma)d\sigma\right) r_{1,i}(s)ds \tag{85}$$

with

$$\lambda^i(0) = \exp\left(\int_0^T F_1(s)ds\right)\int_0^T \exp\left(-\int_s^T F_1(\sigma)d\sigma\right) r_{1,i}(s)ds. \tag{86}$$

In this example, the static game (37) satisfies for each player $i \in N$ and $t \in [0,T]$

$$G^i(t, \mathbf{a}, \lambda^i(t)) = \mathbf{a}^\star Q_i(t)\mathbf{a} + \lambda^i(t)F_2(t, \mathbf{a}). \tag{87}$$

From Theorem 3.1 and Theorem 3.2, $\bar{\mathbf{a}}(\cdot)$ is a Nash equilibrium for (84) if and only if for all t

$$\frac{\partial}{\partial a_i}\left(\mathbf{a}^\star Q_i(t)\mathbf{a} + \lambda^i(t)F_2(t,\mathbf{a})\right)|_{\mathbf{a}=\bar{\mathbf{a}}(t)} = 0 \quad \forall i \in N. \tag{88}$$

Therefore, $\bar{\mathbf{a}}(\cdot)$ is a Nash equilibrium for (84) if and only if for all t, $\bar{\mathbf{a}}(t)$ solves the following system of equations

$$R_i\left[Q_i(t) + Q_i(t)^*\right]\bar{\mathbf{a}}(t) + \lambda^i(t)\frac{\partial}{\partial a_i}F_2(t,\mathbf{a})|_{\mathbf{a}=\bar{\mathbf{a}}(t)} = 0 \quad \forall i \in N, \tag{89}$$

where $R_i\left[Q_i(t) + Q_i(t)^*\right]$ is the ith row of $Q_i(t) + Q_i(t)^*$.

As a particular case, if $F_2(t,\mathbf{a}) = D(t)\mathbf{a}$, $D(t) = (D_1(t), \ldots, D_k(t))$, then from (89) $\bar{\mathbf{a}}(\cdot)$ is a Nash-equilibrium for (84) if and only if $\bar{\mathbf{a}}(t)$ solves the following system of linear equations

$$R_i\left[Q_i(t) + Q_i(t)^*\right]\bar{\mathbf{a}}(t) = -\lambda^i(t)D_i(t) \quad \forall i \in N, \tag{90}$$

for all t. ♢

Example 6.3 Consider the stochastic OCP with $X = \mathbb{R}$ and $A = \mathbb{R}^m$ with

$$\begin{cases} \mathrm{d}x(t) = [F_1(t)x(t) + F_2(t, a(t))]\,\mathrm{d}t + \sum_{j=1}^{k} [B_j(t)x(t) + b_i(t)]\,\mathrm{d}W_j(t) \\ r(t, x, a) = r_1(t)x + a^{\star}Q(t)a \end{cases} \tag{91}$$

with F_1, F_2, r_1 as in Section 2, for all $1 \leq j \leq k$, $B_i(\cdot)$, $b_i(\cdot)$ are piecewise-continuous and $\mathbf{W}$ a $d-$dimensional Brownian motion, as in (40).

From equation (53), the optimal control $\bar{a}(\cdot)$ satisfies $P - a.s.$

$$\bar{a}(t)^{\star}Q(t)\bar{a}(t) + \bar{p}(t)F_2(t, \bar{a}(t)) = \max_{a \in A} \{a^{\star}Q(t)a + \bar{p}(t)F_2(t, a)\}\,. \tag{92}$$

Note that (92) is the same as equation (79) with $\bar{p}(t)$ instead of $\lambda(t)$. Therefore, as in Example 6.1, $\bar{a}(\cdot)$ satisfies $P - a.s.$

$$\bar{a}(t) = -\bar{p}(t)\left(Q(t) + Q(t)^{\star}\right)^{-1}\left(\frac{\partial}{\partial a}F_2(t, a)|_{a=\bar{a}(t)}\right)^{\star}.$$

It follows from Proposition 4.1 and Corollary 4.1

$$\bar{p}(t) = \lambda(t) = \int_t^T \exp\left(\int_t^s F_1(\tau)\mathrm{d}\tau\right) r_1(s)\mathrm{d}s \quad \forall t \in [0, T]. \tag{93}$$

Observe that for all $t \in [0, T]$, $\bar{p}(t) = \lambda(t)$ where $\lambda(\cdot)$ is the costate function for the OCP in Example 6.1 (see equation (78)). This observation and equation (92) imply that the optimal control for the stochastic OCP (91) is deterministic and coincides with the optimal control in Example 6.1. Hence we have an example of the certainty equivalence principle [14]. ♢

Example 6.3 can be rewritten into a stochastic differential game (as we did for the deterministic case in Examples 6.1 and 6.2). For this stochastic differential game, the certainty equivalence principle is satisfied (see Remark 5.2 in Section 5) . To avoid being repetitious, we omit the proof.

6.2 **Examples using dynamic programming**

Remark 6.1 (a) Given a real-valued function $(t, x) \mapsto v(t, x)$ on $(0, T) \times \mathbb{R}^n$, v_t denotes the partial derivative of v with respect to t, and v_x is the gradient of v, that is, the row vector $(v_{x_1}, ..., v_{x_n})$ of partial derivatives. The Hessian matrix is $v_{xx} = (v_{x_i x_j})$.

(b) Let us consider the infinite-horizon discounted reward OCP (23)-(24). (The finite-horizon case is similar). Let $v(t, x)$ be a real-valued continuously differentiable function on $[0, \infty) \times X$ that satisfies the HJB (or dynamic programming or Bellman) equation [13,29]

$$\alpha v = v_t + \max_{a \in A}[r(t, x, a) + v_x \cdot F(t, x, a)] \tag{94}$$

for all $(t, x) \in [0, \infty) \times X$, with "terminal condition"

$$e^{-\alpha t}v(t, x^*(t)) \to 0 \quad \text{as} \quad t \to \infty \tag{95}$$

where $x^*(\cdot)$ is the state trajectory corresponding to an optimal control.

(c) For an infinite-horizon *stochastic* OCP as in Section 4.2, the HJB equation is as in (94)-(95) except that the left-hand side of (94) is replaced by

$$\alpha v - \frac{1}{2}\sum_{i,j=1}^{n} v_{x_i x_j} d_{i,j} \tag{96}$$

where the $d_{i,j}$ are the components of the matrix $D(x) := \sigma(x)\sigma(x)^*$, whereas $v(t, x^*(t))$ in (95) is replaced by its expected value (see, for instance, Hernandez-Lerma et al. [13]).

The following example is a version of Section 7.12.2 in Haurie et al. [12].

Example 6.4 We wish to maximize

$$J(x, a(\cdot)) := \int_0^\infty e^{-\alpha t}[x(t) + \ln(a(t))]\mathrm{d}t$$

with $0 < \alpha \leq \frac{1}{4}$ and $0 < a(\cdot) < 1$, subject to

$$\dot{x}(t) = (1 - a(t))a(t), \quad x(0) = x > 0. \tag{97}$$

In this case, the HJB equation (95) becomes

$$\alpha v = \max_a [x + \ln(a) + v_x \cdot (1 - a)a],$$

which, taking $v(x) = Mx + N$ as in (73), reduces to

$$\alpha(Mx + N) = x + \max_a [\ln(a) + M(1 - a)a].$$

Comparing coefficients in this equation we obtain $M = \frac{1}{\alpha}$, whereas the right-hand side is maximized by the positive root, say a^*, of the quadratic equation

$$2Ma^2 - Ma - 1 = 0, \quad i.e., \quad a^* = \frac{1}{4}(1 + \sqrt{(}1 + 8\alpha)).$$

Note that the *constant* optimal control $\bar{a}(\cdot) \equiv a^*$ is in $(0, 1)$. Moreover, $N = (\frac{1}{\alpha})[\ln a^* + M(1 - a^*)a^*]$. Finally, from (97) we obtain the state trajectory $\bar{x}(\cdot)$ corresponding to $\bar{a}(\cdot)$, and then the condition (95), i.e., $e^{-\alpha t}v(\bar{x}(t)) \to 0$ is trivially satisfied. ◊

Example 6.5 Consider the following stochastic OCP: maximize

$$E\left[\int_0^\infty (\beta x(t) + a(t)^\gamma)e^{-\rho t}\right] dt,$$

with $\beta > 0, \gamma \in (0, 1)$ and $a(\cdot) \geq 0$, subject to

$$dx(t) = (\delta x(t) - a(t))dt + \sigma x(t)dW(t), \quad x(0) = x > 0 \tag{98}$$

with coefficients $0 < \delta < \rho$ and $\sigma > 0$. Suppose that the value function $v(\cdot)$ is as in (73). Then, with $v(x) = Mx + N$, the HJB equation $(94) - (96)$ becomes

$$\rho(Mx + N) = (\beta + M\delta)x + \max_a (a^\gamma - Ma).$$

Hence, comparing coefficients, we see that $M = \frac{\beta}{\rho - \delta}$, and the maximum at the right-hand side is attained at $a^* = \left(\frac{\gamma}{M}\right)^{\frac{1}{1-\gamma}}$. Moreover, $N = \rho^{-1}((a^*)^\gamma - Ma^*)$. Finally, it remains to verify (95). To this end, replacing the constant optimal control $\bar{a}(\cdot) \equiv a^*$ in (98) we obtain that the state trajectory satisfies the stochastic linear equation

$$dx = (\delta x - a^*)dt + \sigma x dW, \quad x(0) = x.$$

Therefore $\bar{x}(\cdot) \equiv x(\cdot)$ is given by (see Arnold [1], for instance)

$$x(t) = \left(x - \frac{a^*}{\delta}\right) e^{\delta t} + \frac{a^*}{\delta} + \sigma \int_0^t x(s) e^{\delta(s-t)} dW(s).$$

Therefore, for any initial state $x(0) = x$, $e^{-\rho t} E[v(x(t))] = e^{-\rho t}(ME[x(t)] + N) \to 0$ as $t \to 0$ since, by assumption, $\rho > \delta$. $\Diamond$

Remark 6.2 A key hypothesis in our results is that $r_1(\cdot) \not\equiv 0$. If this is not true, then our approach might fail. For instance, by (19)-(21), the adjoint function $\lambda(\cdot)$ vanishes if $r_1(\cdot) \equiv 0$. Hence, as illustrated in the following Examples 6.6 and 6.7, the OCP's value function V is not necessarily linear.

Example 6.6 (Exercise 7 in Dockner et al. 2000, page 82). The OCP is to maximize

$$\int_0^\infty e^{-\alpha t} a^\gamma(t) \mathrm{d}t$$

subject to $\dot{x}(t) = x(t) - a(t)$, with $x(\cdot)$ and $a(\cdot) \geq 0$, and $x(0) = x_0 > 0$. This is an OCP as in Section 2 with state transition function and running reward $F(t, x, a) = x - a$ and $r(t, x, a) = a^\gamma$, respectively. The corresponding HJB equation is given by

$$v(x,t) - v_t(x,t) = \max_a \{a^\gamma - v_x(x,t)a\} + v_x(x,t)x \tag{99}$$

Let

$$d_\gamma(\alpha) := \max_a \{a^\gamma - \alpha a\} \quad \forall \alpha > 0, \tag{100}$$

and let $a_\gamma(\alpha)$ be the point where $a \mapsto a^\gamma - \alpha a$ attains its maximum, that is

$$a_\gamma(\alpha) = \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{1-\gamma}} \quad \forall \alpha > 0. \tag{101}$$

Therefore

$$d_\gamma(\alpha) = \alpha \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{1-\gamma}} \left(\frac{1-\gamma}{\gamma}\right). \tag{102}$$

Note that for all $\alpha > 0$, $v_1(x, t; \alpha) := \alpha x + d_\gamma(\alpha)$ satisfies equation (99), and for this case, we get

$$\bar{a}(t; \alpha) = a_\gamma(\alpha) \quad \forall t \geq 0,$$

and

$$\bar{x}(t; \alpha) = e^t \left(x_0 - a_\gamma(\alpha) \int_0^t e^{-s} ds\right) \quad \forall t \geq 0.$$

More explicit,

$$\bar{x}(t; \alpha) = e^t (x_0 - a_\gamma(\alpha)) + a_\gamma(\alpha) \quad \forall t \geq 0.$$

Note that we need for all $t \geq 0$, $\bar{x}(t; \alpha) \geq 0$, so it is necessary that

$$x_0 - a_\gamma(\alpha) \geq 0. \tag{103}$$

Now from Lemma 3.1(i) in [8] (see page 64), $v_1(x, t; \alpha)$ is the value function of the OCP (optimality is understood in the sense of the catching up criterion, see [8] page 63) if

$$\limsup_{t\to\infty} e^{-t} v_1(x^\star(t), t; \alpha) \leq 0,$$

that is, if

$$x_0 - a_\gamma(\alpha) \leq 0. \tag{104}$$

There is only one α which satisfies (103) and (104); this is given by $\alpha_0 = \frac{\gamma}{x_0^{1-\gamma}}$.

Therefore,

$$v_1\left(x, t; \frac{\gamma}{x_0^{1-\gamma}}\right) = \left(\frac{\gamma}{x_0^{1-\gamma}}\right) x + d_\gamma\left(\frac{\gamma}{x_0^{1-\gamma}}\right), \tag{105}$$

where

$$d_\gamma\left(\frac{\gamma}{x_0^{1-\gamma}}\right) = \gamma x_0^\gamma \left(\frac{1-\gamma}{\gamma}\right). \tag{106}$$

From equations (105) and (106), the *value* of the OCP is given by

$$V(x_0) = v_1\left(x_0, 0; \frac{\gamma}{x_0^{1-\gamma}}\right) = x_0^\gamma \quad \forall x_0 > 0.$$

This implies that V is not linear in the initial conditions as in Theorem 2.4. Note that the optimal control and trajectory are given by

$$\bar{a}(t; \alpha_0) = x_0 \quad \bar{x}(t; \alpha_0) = x_0 \quad \forall t \geq 0,$$

respectively. $\Diamond$

Example 6.7 (Example 6.15 in Hernández–Lerma et al. [13]). Maximization of total discounted utility of consumption.

$$\mathrm{d}x(t) = [\alpha x(t) - a(t)]\mathrm{d}t + \sigma x(t)\mathrm{d}W(t)$$

with $x(0) = x > 0$. The OCP is to maximize the expected discounted utility

$$E\left[\int_0^\infty e^{-\rho t} U[a(t)]\mathrm{d}t\right],$$

where $U(\cdot)$ is a given utility function. Since $r_1(\cdot) \equiv 0$, the value function $V(\cdot)$ might not be linear in the state. For instance, if $U(a) = \frac{a^\gamma}{\gamma}$, with $0 < \gamma < 1$, then $V(x) = hx^\gamma$ for some constant $h > 0$. For details, see [13], Example 6.15.

7 Conclusions

In this paper we have studied a class of *linear-state* optimal control problems and noncooperative differential games. Deterministic and stochastic systems were considered, as well as finite- and infinite-horizon problems.

By means of the *maximum principle* [2,11,13,27,29], we showed that the optimal control is a function of the time parameter t only, which means that the optimal control is *independent of the state process*. This fact implies that (i) the OCP's value function is *linear in the state variable* and, in addition, (ii) for the stochastic case, the *certainty equivalence principle* is satisfied.

Throughout this paper, we assume that the reward function r (see (5) and (41)) depends on the state variable, that is $r_1 \not\equiv 0$. If this does not hold, we might not obtain the linearity of the value function (See Remark 6.2 and Examples 6.6 and 6.7). In this case, equations (21), (27), (52) and Lemma 4.1 imply that $\lambda \equiv 0$ and $\bar{p} \equiv 0$, which means that the optimal control, $\bar{a}(t)$, satisfies

$$r_2(t, \bar{a}(t)) = \max_{a \in A} \{r_2(t, a)\}. \tag{107}$$

From equation (107), we get that when r_2 is time-invariant, the optimal control solves an *optimization problem*. OCPs with this property are called myopic, see for instance [25].

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Appendix: proof of Corollaries 2.1 and 2.2

Proof of Corollary 2.1 By Theorem 2.1

$$I = \Phi(t, s)\Phi(s, t),$$

so

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, s)\Phi(s, t) = \frac{\mathrm{d}\Phi(t, s)}{\mathrm{d}t}\Phi(s, t) + \Phi(t, s)\frac{\mathrm{d}\Phi(s, t)}{\mathrm{d}t}.$$

Hence

$$\Phi(t, s)\frac{\mathrm{d}\Phi(s, t)}{\mathrm{d}t} = -\frac{\mathrm{d}\Phi(t, s)}{\mathrm{d}t}\Phi(s, t) = -M(t)\Phi(t, s)\Phi(s, t),$$

$$\frac{\mathrm{d}\Phi(s, t)}{\mathrm{d}t} = -\Phi(s, t)M(t).$$

□

Proof of Corollary 2.2 Let $y : [a, b] \to \mathbb{R}^{n\times n}$ be the unique solution to the initial value problem

$$\dot{y}(t) = \hat{M}(t)y(t) \quad y(0) = y_0,$$

and let $x : [a, b] \to \mathbb{R}^{n\times n}$ be the unique solution to the initial value problem

$$\dot{x}(t) = M(t)y(t) \quad x(0) = x_0.$$

Then

$$\frac{\mathrm{d}\langle x, y\rangle}{\mathrm{d}t} = \langle \dot{x}, y\rangle + \langle x, \dot{y}\rangle,$$

and

$$\langle \dot{x}, y\rangle + \langle x, \dot{y}\rangle = \langle Mx, y\rangle + \langle x, -M^{\star}y\rangle.$$

Hence

$$\langle Mx, y\rangle + \langle x, -M^\star y\rangle = \langle x, M^\star y\rangle + \langle x, -M^\star y\rangle.$$

It follows that, for all $t \in [a,b]$, $\frac{\mathrm{d}\langle x,y\rangle}{\mathrm{d}t} = 0$.

Therefore, there exists $c \in \mathbb{R}$ such that for all $t \in [a,b]$, $\langle x(t), y(t)\rangle = c$.

Since $x(t) = \Phi(t,a)x_0$ and $y(t) = \hat{\Phi}(t,a)y_0$,

$$\langle \Phi(t,a)x_0, \hat{\Phi}(t,a)y_0\rangle = c \quad \forall t \in [a,b].$$

In particular, if $t = a$, then $c = \langle x_0, y_0\rangle$. Note that for all $t \in [a,b]$,

$$\langle \Phi(t,a)x_0, \hat{\Phi}(t,a)y_0\rangle = \langle x_0, \Phi^\star(t,a)\hat{\Phi}(t,a)y_0\rangle.$$

It follows that, for all $t \in [a,b]$,

$$\langle x_0, \Phi^\star(t,a)\hat{\Phi}(t,a)y_0\rangle = \langle x_0, y_0\rangle,$$

and $\langle x_0, \left(\Phi^\star(t,a)\hat{\Phi}(t,a) - I\right) y_0\rangle = 0$. Since we can let y_0 be arbitrary, we conclude that

$$\Phi^\star(t,a)\hat{\Phi}(t,a) = I \quad \forall t \in [a,b],$$

that is, $\hat{\Phi}(t,a) = (\Phi^\star)^{-1}(t,a) = \Phi^\star(a,t)$ by Theorem 2.1. □

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